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A Bound for the Pressure Integral in a Plasma Equilibrium

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An interpolation inequality for the total variation of the gradient of a composite function is derived by applying the coarea formula. A bound for the pressure integral is studied by establishing an *a priori* estimate for a solution of the Grad–Shafranov equation of plasma equilibrium. A weak formulation of the Grad–Shafranov equation is given to include singular current profiles.

KEY WORDS: Plasma equilibrium; Grad–Shafranov equation; interpolation inequality.

1. INTRODUCTION

A simple but essential question in the fusion plasma research is how large a plasma energy can be confined by a given magnitude of plasma current.[9,21–23] In a magnetohydrodynamic equilibrium of a plasma, the thermal pressure force ∇p is balanced by the magnetic stress $\mathbf{j} \times \mathbf{B}$, where $\mathbf{B}$ is the magnetic flux density, $\mathbf{j} = \nabla \times \mathbf{B}/\mu_0$ is the current density in the plasma, and $\mu_0 = 4\pi \times 10^{-7}$ is the vacuum permeability. The plasma equilibrium equation $\nabla p = \mathbf{j} \times \mathbf{B}$ thus relates the pressure and the current. We want to estimate the maximum of the total pressure with respect to a fixed total current. Mathematically this problem reduces to an *a priori* estimate for the pressure integral with respect to a solution of the equilibrium equation with a given magnitude of current.

Here we assume a simple two-dimensional plasma equilibrium. Let $\Omega \subset \mathbf{R}^2$ be a bounded domain. We consider an infinitely long plasma column; Ω corresponds to the cross section of a column containing the

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plasma. If there is no longitudinal magnetic field, the equilibrium equations are

$$-\Delta\Psi = P'(\psi) \qquad \text{in } \Omega, \tag{1.1}$$

$$\psi = c \qquad \text{on } \partial\Omega \tag{1.2}$$

$$\int_\Omega (-\Delta\psi)\, dx = \mu_0 I, \tag{1.3}$$

where ψ is the flux function, $P = \mu_0 p$, $P(t)$ is a nonnegative function from $\mathbf{R}$ to $\mathbf{R}$, $P' = dP(t)/dt$, I is a given positive constant, and c is an unknown constant; see Section 3.1 for the derivation, and also see refs. 9, 10, 17, and 21. We assume $P' \geqslant 0$. Since $-\Delta\psi/\mu_0$ parallels the current density, I represents the total plasma current. The total pressure in a unit length of the plasma column is given by integrating p over Ω. In this paper we study a bound for the (poloidal) beta ratio,[9,23] which is defined by

$$\beta = \int_\Omega p\, dx/(I^2\mu_0/8\pi) = 8\pi \int_\Omega P(\psi)\, dx \bigg/ \left(\int_\Omega (-\Delta\psi)\, dx\right)^2 \tag{1.4}$$

A crucial step is to establish an interpolation inequality to estimate the total variation of the gradient of $P(\psi)$ in Ω. Our estimate reads

$$\int_\Omega |\nabla P(\psi(x))|\, dx \leqslant 2\left(P_{\max} \int_\Omega -\Delta\psi\, dx\right)^{1/2} \left(\int_\Omega P'(\psi(x))\, dx\right)^{1/2} \tag{1.5}$$

provided that $-\Delta\psi \geqslant 0$ in Ω and $\psi = c$ on $\partial\Omega$, and that $P' \geqslant 0$ with $P(c) = 0$, where c is a constant and $P_{\max}$ is the maximum of $P(\psi)$ over Ω. We prove this estimate by using the coarea formula.[8,13] Using the Hölder and isoperimetric inequalities, one obtains the estimate for β:

$$\beta \leqslant 8/\alpha \tag{1.6}$$

where $\alpha = S^*/S_0$, S_0 is the area of the support of $P(\psi)$ in Ω, and

$$S^* = \int_\Omega P(\psi(x))\, dx/P_{\max}$$

We include the situation when P is not continuous. In this case the meaning of the equation $-\Delta\psi = P'(\psi)$ is not clear. We shall give a meaning for discontinuous P and prove (1.6) for such a P. In Section 2 we prove (1.5) and extend it for discontinuous P. In Section 3 we briefly review the plasma equilibrium equations (1.1)–(1.3). Together with a mathematical formulation of the equations for discontinuous P, we derive a bound (1.6) for β.

Let us concisely survey related theories. For a given profile of $P(\psi)$, the equilibrium equations (1.1)–(1.3) are regarded as a nonlinear eigenvalue problem. A simple example is to take $P'(\psi) = \lambda\psi^+$, where $\lambda \in \mathbf{R}$ and $\psi^+(x) = \max\{\psi(x), 0\}$. Then the boundary of the support of ψ^+ is a free boundary. Analytic aspects of such a free boundary problem were extensively studied by many authors; see, e.g., refs. 1, 4, 11, 15, and 18–20 and references therein. In ref. 3 a more general problem including the effect of external control field is studied, while the relation of P and ψ is still given by $P'(\psi) = \lambda\psi^+$. In ref. 2 an inverse problem determining P' of (1.1) is studied by measuring $\partial\psi/\partial n$ on $\partial\Omega$. Our *a priori* estimate (1.6) for β as well as the interpolation inequality (1.5) seem to be new even for a smooth P.

In the physics literature, however, there have been many discussions on the limitation of β. A mostly simple configuration is a circular-cross-section z-pinch equilibrium (plasma column with only poloidal field; see Section 3.1), which was studied by Bennett to show $\beta = 1$ for any (smooth) profile of $P(\psi)$; see Chapter 5 of ref. 9 and Remark 3.5. Studies on a toroidal equilibrium such as a tokamak (e.g., refs. 9 and 21) are of principal importance. The high-poloidal-beta regime of tokamak equilibria is attracting much interest because of many beneficial reasons for optimizing the fusion plasma confinement. A bound for the poloidal beta was estimated by using approximate equilibrium solutions[5,6,22,23] and numerical calculations[7]; however, there is no rigorous conclusion. To study a toroidal plasma equilibrium, one should use a generalized toroidal version of the Grad–Shafranov equation[10,17] instead of the simplified one (1.1), in order to include the toroidal curvature effect as well as the theta-pinch effect in addition to the z-pinch effect described by (1.1). Our assertion in Section 3.2 is restricted to straight z-pinches; however, the method of Section 2 may be useful for the analysis of the tokamak problem.

2. AN INTERPOLATION INEQUALITY

Our goal in this section is to estimate the total variation of $\nabla(P(\psi))$ (as a vector-valued measure), where P is monotone and $-\Delta\psi \geqslant 0$. We first derive the estimate for smooth ψ.

Theorem 2.1. Let Ω be a bounded domain in $\mathbf{R}^n$ ($n \geqslant 1$) and c be a constant. Suppose that $P \in C^1(\mathbf{R})$ with $P' \geqslant 0$ and $P(c) = 0$, and that $\psi \in C^m(\Omega) \cap C^0(\bar{\Omega})$ with

$$\begin{aligned} -\Delta\psi &\geqslant 0 \quad && \text{in } \Omega, \\ \psi &= c \quad && \text{on } \partial\Omega \end{aligned} \tag{2.1}$$

where $m \geqslant 2$ and $m \geqslant n$. Let $P_{\max}$ denote

$$P_{\max} = \sup_{x \in \Omega} P(\psi(x)) \tag{2.2}$$

Then

$$\int_\Omega |\nabla P(\psi(x))| \, dx \leqslant 2 \left(P_{\max} \int_\Omega (-\Delta\psi) \, dx \right)^{1/2} \left(\int_\Omega P'(\psi(x)) \, dx \right)^{1/2} \tag{2.3}$$

Proof. If $-\Delta\psi \equiv 0$, then $\psi = c$ on Ω, so (2.3) holds with zero for both sides. If $P'(\psi) \equiv 0$ on Ω or $P_{\max} = 0$, then either $\psi \equiv c$ or $P \equiv 0$. Again (2.3) holds in this case, so we may assume that both integrals in the right-hand side of (2.3) are nonzero. We may also assume that the L^1 norm of $-\Delta\psi$ is finite.

For $K > 0$ denote the set of $x \in \Omega$ for which $|\nabla\psi(x)| > K$ by D. Let E denote the complement of D in Ω. From the definition it follows that

$$\begin{aligned} \int_E |\nabla P(\psi(x))| \, dx &= \int_E P'(\psi) |\nabla\psi| \, dx \\ &\leqslant K \int_E P'(\psi) \, dx \leqslant K \int_\Omega P'(\psi) \, dx \end{aligned} \tag{2.4}$$

since $P' \geqslant 0$.

By applying the maximum principle to (2.1), we observe that $\psi \geqslant c$ on Ω, so $0 = P(c) \leqslant P(\psi) \leqslant P_{\max}$ on Ω. Applying the coarea formula (see, e.g., refs. 8, 13) yields

$$\int_D |\nabla P(\psi)| \, dx = \int_{-\infty}^{+\infty} \mathscr{H}^{n-1}(S_t) \, P'(t) \, dt = \int_c^{\psi_{\max}} \mathscr{H}^{n-1}(S_t) \, P'(t) \, dt \tag{2.5}$$

with

$$S_t = D \cap L_t, \qquad L_t = \{x \in \Omega; \psi(x) = t\}, \qquad \psi_{\max} = \sup_{x \in \Omega} \psi(x)$$

where $\mathscr{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure. Since $|\nabla\psi| > K$ on D, it follows that

$$\mathscr{H}^{n-1}(S_t) = \int_{S_t} |\nabla\psi| \cdot |\nabla\psi|^{-1} \, d\mathscr{H}^{n-1} \leqslant K^{-1} \int_{L_t} |\nabla\psi| \, d\mathscr{H}^{n-1}$$

Since $\psi \in C^n(\Omega)$, Sard's theorem[12] implies that L_t is a C^n submanifold in Ω for almost every t (a.e. t). Note that $\psi > c$ in Ω and $\psi = c$ in Ω and $\psi = c$ on $\partial\Omega$. Thus for $U_t = \{x \in \Omega; \psi(x) > t\}$ we observe $\bar{U}_t \subset \Omega$ for $t > c$. For a.e. $t>c$, L_t is a C^n boundary of U_t. Since L_t is a t-level set of ψ, $\mathbf{n}=\nabla\psi/|\nabla\psi|$ is a unit normal vector field. Applying Green's formula yields

$$\int_{L_t} |\nabla\psi| \, d\mathscr{H}^{n-1} = \int_{L_t} \nabla\psi \cdot \mathbf{n} \, d\mathscr{H}^{n-1} = \int_{U_t} (-\Delta\psi) \, dx, \qquad t>c$$

From $-\Delta\psi \geqslant 0$ it now follows that

$$\int_{L_t} |\nabla\psi| \, d\mathscr{H}^{n-1} \leqslant \int_{\Omega} (-\Delta\psi) \, dx$$

Wrapping up these two estimates, we obtain

$$\mathscr{H}^{n-1}(S_t) \leqslant K^{-1} \int_{\Omega} (-\Delta\psi) \, dx$$

Applying this estimate to (2.5) yields

$$\int_{D} |\nabla P(\psi)| \, dx \leqslant K^{-1} P_{\max} \int_{\Omega} (-\Delta\psi) \, dx \tag{2.6}$$

where $P_{\max}$ is defined in (2.2). Summing (2.4) and (2.6), we obtain

$$\int_{\Omega} |\nabla P(\psi)| \, dx \leqslant K \int_{\Omega} P'(\psi) \, dx + K^{-1} P_{\max} \int_{\Omega} (-\Delta\psi) \, dx \tag{2.7}$$

for arbitrary $K>0$. Taking

$$K = \left[P_{\max} \int_{\Omega} (-\Delta\psi) \, dx \Big/ \int_{\Omega} P'(\psi) \, dx \right]^{1/2}$$

in (2.7) yields (2.3). ∎

If ψ is not C^2, one should interpret $-\Delta\psi \geqslant 0$ in the distribution sense. As is well known,[16] a nonnegative distribution is a nonnegative Radon measure. Let μ be a finite Radon measure on a bounded domain Ω in $\mathbf{R}^n$. The unique solvability of the Dirichlet problem

$$-\Delta\psi = \mu \qquad \text{in } \Omega, \tag{2.8a}$$

$$\psi = c \qquad \text{on } \partial\Omega \ (c \text{ constant}) \tag{2.8b}$$

is now well known for a smooth boundary $\partial\Omega$. We solve this problem by using a result of Simader[14] when the boundary is C^1. Let $W^{1,q}(\Omega)$ denote the L^q Sobolev space of order one ($1<q<\infty$). Let $W_0^{1,q}(\Omega)$ be the subspace $\{u \in W^{1,q}(\Omega);\ u=0 \text{ on } \partial\Omega\}$. We denote by $W^{-1,q}(\Omega)$ the dual space of $W_0^{1,q'}(\Omega)$ where $1/q = 1 - 1/q'$.

Lemma 2.2 (Theorem 4.6 of Simader[14]). Let Ω be a bounded domain with C^1 boundary in $\mathbf{R}^n$. Assume that $1 < q < \infty$. For each $f \in W^{-1,q}(\Omega)$ there is a unique solution $\Phi \in W_0^{1,q}(\Omega)$ for $-\Delta\Phi = f$ in Ω. Moreover, the mapping from f to Φ is bounded linear from $W^{-1,q}(\Omega)$ to $W_0^{1,q}(\Omega)$, i.e.,

$$\|\Phi\|_{1,q} \leqslant C \|f\|_{-1,q} \tag{2.9}$$

with a constant $C = C(\Omega, q, n)$.

Corollary 2.3. Let Ω be a bounded domain with C^1 boundary in $\mathbf{R}^n$. For a finite Radon measure μ on Ω there is a unique solution ψ of (2.8a), (2.8b) such that $\psi \in W^{1,r}(\Omega)$ for $1 < r < n/(n-1)$.

Proof. Observe that $r' > n$ implies $W_0^{1,r'}(\Omega) \subset C(\bar{\Omega})$ by the Sobolev inequality. This yields $\mu \in W^{-1,r}(\Omega)$ by a duality, where $1/r = 1 - 1/r'$. Applying Lemma 2.2 with $f = \mu$ obtains a unique solution ψ by $\psi = \Phi + c$. ∎

Theorem 2.4. Let Ω be a bounded domain with C^1 boundary in $\mathbf{R}^n$. Let c be a constant. Suppose that $P \in C^1(\mathbf{R})$ with $P' \geqslant 0$ and $P(c) = 0$. Suppose that $\psi \in W^{1,r}(\Omega)$ for some r such that $1 < r < n/(n-1)$, and that ψ satisfies

$$\begin{aligned} -\Delta\psi &\geqslant 0 \qquad \text{in } \Omega \text{ (in the distribution sense)} \\ \psi &= c \qquad \text{on } \partial\Omega \end{aligned}$$

Let $\psi_{\max}$ be the essential supremum of ψ over Ω. Assume that P and P' are bounded on $[c, \psi_{\max})$. Then

$$\int_\Omega |\nabla P(\psi(x))|\, dx \leqslant 2(P_{\max} \|-\Delta\psi\|_1)^{1/2} \left(\int_\Omega P'(\psi(x))\, dx\right)^{1/2} \tag{2.10}$$

where $P_{\max} = \sup\{P(\sigma);\ c \leqslant \sigma \leqslant \psi_{\max}\}$ and $\|\cdot\|_1$ denotes the total variation of a measure on Ω.

Proof. We may assume that ψ and P are nonconstants and that $-\Delta\psi = \mu$ is a nonnegative finite Radon measure on Ω. Modifying $P(\sigma)$ for $\sigma > \psi_{\max}$, we may assume $\bar{P} = \sup\{P(\sigma);\ \sigma \geqslant c\}$ is very close to $P_{\max}$, say $\bar{P} \leqslant P_{\max} + \varepsilon$, $\varepsilon > 0$.

We extend μ outside Ω by zero and define $\mu_l = \mu * \rho_l$, where ρ_l is Friedrichs' mollifier such that ρ_l tends to the delta function as $l \to \infty$.

Let $\psi_l \in W^{1,q}(\Omega)$ be the solution of (2.8a), (2.8b) with $\mu = \mu_l$. Since μ_l is bounded, Lemma 2.2 implies that $\psi_l - c \in W_0^{1,q}(\Omega)$ for all $q > 1$. By the Sobolev inequality we see that $\psi_l \in C(\bar{\Omega})$. Since μ_l is smooth, the interior regularity of the Poisson equation implies that $\psi_l \in C^\infty(\Omega)$. Theorem 2.1 yields

$$\int_\Omega |\nabla P(\psi_l)|\, dx \leqslant 2\left(\bar{P}\int_\Omega (-\Delta\psi_l)\, dx\right)^{1/2}\left(\int_\Omega P'(\psi_l)\, dx\right)^{1/2} \qquad (2.11)$$

Since $\mu \in W^{-1,r}(\Omega)$ is expressed as

$$\mu = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} f_j + g$$

with some f_j, $g \in L^r(\Omega)$, we have $\mu_l \to \mu$ strongly in $W^{-1,r}(\Omega)$. The inequality (2.9) thus implies that $\psi_l \to \psi$ in $W^{1,r}(\Omega)$. We may assume $\psi_l(x) \to \psi(x)$ and $\nabla\psi_l(x) \to \nabla\psi(x)$ for a.e. x by taking a subsequence if necessary. Applying the Lebesgue dominated convergence theorem yields

$$\int_\Omega |\nabla P(\psi_l)|\, dx \to \int_\Omega |\nabla P(\psi)|\, dx$$

$$\int_\Omega P'(\psi_l)\, dx \to \int_\Omega P'(\psi)\, dx$$

since P' is bounded on $[c, +\infty)$. Clearly,

$$\int_\Omega (-\Delta\psi_l)\, dx \to \int_\Omega (-\Delta\psi)\, dx = \|-\Delta\psi\|_1$$

Letting $l \to \infty$ in (2.11) yields

$$\int_\Omega |\nabla P(\psi(x))|\, dx \leqslant 2(\bar{P}\|-\Delta\psi\|_1)^{1/2}\left(\int_\Omega P'(\psi(x))\, dx\right)^{1/2}$$

Since $\bar{P} \leqslant P_{\max} + \varepsilon$ and $\varepsilon > 0$ can be chosen arbitrary, this leads to (2.10). ∎

We next extend the inequality (2.10) when a nondecreasing function P is not necessarily continuous. Let us give an interpretation of each integral appearing in (2.10). Instead of the integral $\int_\Omega P'(\psi)\, dx$, we consider

$$[P'(\psi)] = \inf \varliminf_{l \to \infty} \int_\Omega P_l'(\psi)\, dx$$

Here the infimum is taken over all sequences $P_l \in C^1(\mathbf{R})$ with $P_l' \geqslant 0$ such that $P_l(\psi) \to P(\psi)$ in $L^s(\Omega)$ for some $1 \leqslant s < \infty$ as $l \to \infty$ and that $(P_l)_{\max} \to \operatorname{ess\,sup}_\Omega P(\psi)$. We say $\{P_l\}$ is an *admissible approximation* of P if these properties hold. If P is itself C^1 and satisfies the assumptions in Theorem 2.4, P itself is an admissible approximation, so for such a P we have

$$[P'(\psi)] \leqslant \int_\Omega P'(\psi)\, dx$$

Since $\int_\Omega |\nabla P(\psi)|\, dx$ is the total variation of $\nabla P(\psi)$ on Ω, i.e.,

$$\|\nabla P(\psi)\|_1 = \int_\Omega |\nabla P(\psi(x))|\, dx$$

$$:= \sup\left\{\int_\Omega P(\psi(x))\, \nabla \cdot \varphi(x)\, dx;\ \varphi \in C_0^1(\Omega),\ |\varphi(x)| \leqslant 1 \text{ on } \Omega\right\}$$

it is easy to see that

$$\|\nabla P(\psi)\|_1 \leqslant \varliminf_{l \to \infty} \int_\Omega |\nabla P_l(\psi)|\, dx$$

for any admissible approximation $\{P_l\}$ of P since $\sup \varliminf \leqslant \varliminf \sup$. We have thus proved the following assertion.

Theorem 2.5. Assume the hypotheses of Theorem 2.4 concerning c, Ω, and ψ. Let P be a nondecreasing function on $\mathbf{R}$ with $P(c) = 0$. Then

$$\|\nabla P(\psi)\|_1 \leqslant 2(P_{\max} \|-\Delta\psi\|_1)^{1/2} [P'(\psi)]^{1/2} \tag{2.12}$$

provided that $P_{\max} = \operatorname{ess\,sup}_\Omega P(\psi)$ is finite.

Remark. If $P(\sigma) = \sigma$, the inequality (2.10) is an interpolation inequality

$$\|\nabla\psi\|_1 \leqslant 2(P_{\max} \|-\Delta\psi\|_1)^{1/2} |\Omega|^{1/2}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω.

3. APPLICATION TO PLASMA PHYSICS; ESTIMATE OF THE BETA RATIO

3.1. Background of the Problem

In this section, we describe an application of the interpolation inequality derived in Section 2. We study the upper bound of the plasma pressure integral for a given magnitude of plasma current. We consider a plasma equilibrium where the pressure force ∇p is balanced by the magnetic force $\mathbf{j} \times \mathbf{B} = (\nabla \times \mathbf{B}) \times \mathbf{B}/\mu_0$. Let us briefly review the physical formulation of the plasma equilibrium equations (1.1)–(1.3).

When a plasma equilibrium has an ignorable coordinate, the equilibrium equation

$$(\nabla \times \mathbf{B}) \times \mathbf{B}/\mu_0 = \nabla p \tag{3.1}$$

reduces to a simple nonlinear equation, which is called the Grad–Shafranov equation.[10,17] In this paper, we assume an infinitely long plasma column with $\partial/\partial z = 0$ in the Cartesian coordinates (x_1, x_2, z). Moreover, we consider a simple z-pinch configuration, where the current density vector $\mathbf{j}$ has only the longitudinal z component j_z and $\mathbf{B}$ has only the transverse x and y components (poloidal field). Since $\nabla \cdot \mathbf{B} = 0$, we may write

$$\mathbf{B} = \nabla\psi \times \nabla z \tag{3.2}$$

where $\psi = \psi(x_1, x_2)$ is the flux function. We obtain

$$\nabla \times \mathbf{B} = (-\Delta\psi)\,\nabla z = \mu_0\, j_z\, \nabla z \tag{3.3}$$

Substituting (3.2) and (3.3) into (3.1), we observe that $\nabla\psi$ parallels ∇p, so p is constant on each level set of ψ, i.e., $p = p(\psi)$ formally. The equilibrium equation (3.1) now leads to a (nonlinear) elliptic partial differential equation for ψ,

$$-\Delta\psi = P'(\psi) \qquad \text{in } \Omega \tag{3.4}$$

where $P = \mu_0 p$ and $\nabla P = P'(\psi)\,\nabla\psi$. We note that the pressure in an equilibrium state should satisfy the relation

$$p(x_1, x_2) = P(\psi(x_1, x_2))/\mu_0 \tag{3.5}$$

Here, $P(\psi)$ $(\geqslant 0)$ is an arbitrary function that satisfies the following conditions. First, we assume that $P(\psi)$ is a nondecreasing function, so that

$$P'(\psi(x_1, x_2)) \geqslant 0 \qquad \text{in } \Omega \tag{3.6}$$

The support Ω_P of $P(\psi(x_1, x_2))$ is the plasma region, which must be contained in Ω. Therefore, we assume

$$P = 0 \qquad \text{on } \partial\Omega \tag{3.7}$$

The boundary of Ω_P is the plasma free boundary. The boundary condition on ψ is that the normal component of $\mathbf{B}$ vanishes on the boundary $\partial\Omega$, which reads, by (3.2),

$$\psi = c(=\text{const}) \qquad \text{on } \partial\Omega \tag{3.8}$$

We restrict the total current I (a positive constant); by (3.3) the current through the cross section Ω is given by

$$\int_\Omega j_z\, dx = \mu_0^{-1} \int_\Omega (-\Delta\psi)\, dx = I \tag{3.9}$$

Our goal in this paper is to construct an *a priori* estimate with respect to the beta ratio for the solution to (3.4), (3.8), (3.9). For a current-carrying plasma column, the (poloidal) beta ratio is defined by (1.4). Using the relations (3.5), (3.6), and (3.9), we obtain

$$\beta = 8\pi \|P(\psi)\|_1 / \|-\Delta\psi\|_1^2 \tag{3.10}$$

3.2. Mathematical Formulation and Estimate of the Beta Ratio

We shall give a meaning to $-\Delta\psi = P'(\psi)$ when a nondecreasing function P is not continuous and ψ is not smooth.

Definition 3.1. Suppose that $\psi \in W^{1,r}(\Omega)$ for some r, $1 < r < \infty$, and that P is nondecreasing. We say ψ and P satisfy

$$-\Delta\psi = P'(\psi) \qquad \text{in } \Omega$$

if the following properties hold.

(i) $-\Delta\psi \geqslant 0$ on Ω in the distribution sense.

(ii) There is an admissible approximation $\{P_l\}$ such that

$$\lim_{l\to\infty} \int_\Omega [-\Delta\psi - P_l'(\psi)]\, \varphi\, dx = 0$$

for all $\varphi \in C(\bar{\Omega})$.

Theorem 3.2. Let Ω be a bounded domain with C^1 boundary in $\mathbf{R}^n$. Let c be a constant. Assume that P is a nondecreasing function on $\mathbf{R}$ and that $P(c) = 0$. Assume that $\psi \in W^{1,r}(\Omega)$ for some r, $1 < r < n/(n-1)$, and that ψ satisfies

$$\begin{aligned} -\Delta\psi &= P'(\psi) \qquad \text{in } \Omega \text{ (in the sense of Definition 3.1)} \\ \psi &= c \qquad \text{on } \partial\Omega \end{aligned} \tag{3.11}$$

Then

$$\|\nabla P(\psi)\|_1 \leqslant 2P_{\max}^{1/2}\mu_0 I \tag{3.12}$$

where

$$I = \mu_0^{-1}\int_\Omega (-\Delta\psi)\,dx = \mu_0^{-1}\|-\Delta\psi\|_1.$$

Proof. We may assume $P_{\max} < \infty$. By Definition 3.1(ii) with $\varphi \equiv 1$ we observe that

$$[P'(\psi)] \leqslant \lim_{l\to\infty}\int_\Omega P_l'(\psi)\,dx = \int_\Omega (-\Delta\psi)\,dx = \|-\Delta\psi\|_1$$

since $-\Delta\psi \geqslant 0$. The inequality (2.12) yields (3.12). ∎

Example 3.3. Let Ω be a unit disk in the plane, i.e.,

$$\Omega = \{x \in \mathbf{R}^2;\ |x| < 1\}$$

For $m > 0$ and $0 < R < 1$ we consider

$$\psi(x) = \min\{m,\ -a\log|x|\}, \qquad x \in \Omega$$

Here $a = -m/\log R$, so that ψ is continuous across the circle $|x| = R$. If P is a step function such that

$$\begin{aligned} P(\sigma) &= (a/R)^2 && \text{for} \quad \sigma \geqslant m \\ &= 0 && \text{for} \quad \sigma < m \end{aligned}$$

then ψ and P satisfy $-\Delta\psi = P'(\psi)$ in Ω in the sense of Definition 3.1.

Indeed it is easy to see that $\psi \in W^{1,r}(\Omega)$ for some r, $1 < r < \infty$, and that

$$-\Delta\psi = (a/R)\delta_R \qquad \text{in } \Omega$$

where δ_R is the Dirac measure of the circle $|x| = R$, i.e.,

$$\int_\Omega \varphi\,\delta_R = \int_{|x|=R} \varphi\,ds \qquad \text{for} \quad \varphi \in C_0^\infty(\mathbf{R}^2)$$

We then seek an admissible approximation P_l of P such that Definition 3.1(ii) holds. Let $f \in C^1(\mathbf{R})$ be

$$\begin{aligned} f(\sigma) &= 1 && \text{for} \quad \sigma \geqslant 0 \\ &= 0 && \text{for} \quad \sigma \leqslant -1 \end{aligned}$$

such that $f' \geqslant 0$. If P_l is given by

$$P_l(\sigma) = bf((\sigma - m)l) \qquad \text{with} \quad b = (a/R)^2$$

then P_l is an admissible approximation satisfying (ii). To see this, we proceed with

$$\int_\Omega P_l'(\psi)\,\varphi\,dx = b\int_{R<|x|<1} lf'((-a\log|x| - m)l)\,\varphi\,dx, \qquad \varphi \in C_0^\infty(\Omega) \tag{3.13}$$

We observe that

$$\int_R^1 lf'((-a\log r - m)l)\,dr = \frac{R}{a}\int_{-\infty}^0 f'(\tau)\,d\tau = \frac{R}{a} \tag{3.14}$$

by the change of variables

$$\tau = (-a\log r - m)l$$

Since $lf' \geqslant 0$ and the support of the integrand in (3.13) is an annulus shrinking to the circle $|x| = R$ as $l \to \infty$, applying (3.14) to (3.13) yields

$$\int_\Omega P_l'(\psi)\,\varphi\,dx \to b\frac{R}{a}\int_{|x|=R} \varphi\,ds \qquad \text{as} \quad l \to \infty$$

Since $b = (a/R)^2$, $P_l'(\psi)$ satisfies (ii).

For this choice of P and ψ,

$$\|\nabla P(\psi)\|_1 = (a/R)^2\,2\pi R$$

$$P_{\max} = (a/R)^2, \qquad \|-\Delta\psi\|_1 = (a/R)\,2\pi R$$

so evidently (3.12) holds. There is a possibility that the constant 2 in (3.12) can be replaced by a smaller number. But, as this example shows, the constant should be greater than or equal to one.

As an application of Theorem 3.2, we compute the beta ratio β in (3.10) when Ω is a two-dimensional bounded domain. We introduce several quantities:

$$S_0 = \text{area of the support of } P(\psi) \text{ in } \Omega$$

$$S^* = \|P(\psi)\|_1/P_{\max}$$

$$\alpha = S^*/S_0 \qquad (\leqslant 1)$$

Theorem 3.4. Assume the same hypotheses as Theorem 3.2 concerning Ω, c, P, and ψ with $n=2$. Let β be the beta ratio concerning P and ψ satisfying (3.11). Then

$$\beta \leqslant 8\pi(2C_0)^2/\alpha = 8/\alpha \tag{3.15}$$

with $C_0 = 1/2\pi^{1/2}$, provided that I is finite.

Proof. We may assume $\alpha > 0$. By the Hölder inequality and the isoperimetric inequality (see, e.g., refs. 8 and 13) we obtain

$$\|P(\psi)\|_1 \leqslant (S_0)^{1/2} \|P(\psi)\|_{L^2}^{1/2} \leqslant (S_0)^{1/2}\, C_0 \|\nabla P(\psi)\|_1$$

Theorem 3.2 now yields

$$\begin{aligned}\|P(\psi)\|_1 &\leqslant (S_0)^{1/2} C_0 2 P_{\max}^{1/2} \mu_0 I \\ &\leqslant (S_0/S^*)^{1/2}\, 2C_0 \|P(\psi)\|_1^{1/2}\, \mu_0 I\end{aligned}$$

so that

$$\|P(\psi)\|_1 \leqslant (2C_0 \mu_0 I)^2/\alpha$$

From this follows the desired estimate for β. ▮

Remark. Concerning the z-pinch equilibrium discussed in this section, one has Bennett's pinch relation (see, e.g., ref. 9), i.e., $\beta = 1$ for every $P(\psi)$ as far as $\psi = \psi(r)$, where $r = |x|$ is the radial coordinate. This relation is easily derived by integration by parts. We denote by a the radius of the circular plasma cross section, i.e., the radius of the support of $P(\psi(r))$. Since $P(\psi(a)) = 0$, integrating by parts yields

$$\int_0^a \frac{dP(\psi(r))}{dr}\, r^2\, dr = -\pi^{-1} \|P(\psi)\|_1 \tag{3.16}$$

On the other hand, by $dP(\psi(r))/dr = P'\, d\psi/dr$, we have

$$\begin{aligned}\int_0^a \frac{dP(\psi(r))}{dr}\, r^2\, dr &= \int_0^a P'(\psi)\,\frac{d\psi}{dr}\, r^2\, dr \\ &= F(a)\left(a\,\frac{d\psi}{dr}(a)\right) - \int_0^a F(r)\left(\frac{d}{dr}\left(r\,\frac{d}{dr}\psi\right)\right) dr \end{aligned} \tag{3.17}$$

where we define

$$F(r) = \int_0^r P'(\psi(r))\, r\, dr$$

Using (1.1), we see that the integrand of the second term on the right-hand side of (3.17) is equal to $dF(r)^2/2\,dr$. By (1.3), we obtain

$$2\pi \int_0^a -r^{-1} \frac{d}{dr}\left(r \frac{d}{dr} \psi\right) r\,dr = 2\pi \int_0^a P'(\psi(r))\, r\,dr = \mu_0 I$$

which shows that

$$F(a) = a \frac{d\psi}{dr}(a) = \frac{\mu_0 I}{2\pi}$$

Comparing (3.16) and (3.17), we obtain

$$\|P(\psi)\|_1 = (\mu_0 I)^2/8\pi$$

which implies $\beta = 1$.

In this situation, our estimate (3.15) does not yield the best result, since α can be small for a peaked profile of $P(\psi(r))$. This is because we had to use the Hölder inequality to derive (3.15). Our estimate, however, is useful when we consider a more general configuration.

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